MTE 201

Vector Calculus

1. $\vec{F} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$

$\vec{F}$ is conservative given that $\nabla \times \bar{F} = 0$

$$\therefore \nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right]$$
$$-\hat{j}\left[\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right]$$
$$+\hat{k}\left[\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right]$$

$$= (c+1)\hat{i} - (4-a)\hat{j} + (b-2)\hat{k}$$

This equals zero when $c+1 = 0$,

$a-4 = 0$ and

$b-2 = 0$.

$\therefore$ We have $a = 4$, $b = 2$ and $c = -1$

Such that

$$\bar{F} = (x+2y+4z)\hat{i} + (2x-3y-z)\hat{j} + (4x-y+2z)\hat{k}$$

We want to find ϕ such that $\bar{F} = \nabla\phi$, that is:

$$(x+2y+4z)\hat{i} + (2x-3y-z)\hat{j} + (4x-y+2z)\hat{k}$$
$$= \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$\therefore$ The problem is to find ϕ such that

$$\frac{\partial\phi}{\partial x} = x+2y+4z \quad \ldots \quad (1)$$

$$\frac{\partial\phi}{\partial y} = 2x-3y-z \quad \ldots \quad (2)$$

$$\frac{\partial\phi}{\partial z} = 4x-y+2z \quad \ldots \quad (3)$$

Now, from (1): $\phi = \frac{x^2}{2} + 2xy + 4xz + f(y,z)$

from (2): $\phi = 2xy - \frac{3y^2}{2} - zy + h(x,z)$

from (3): $\phi = 4xz - yz + z^2 + g(x,y)$

$$\therefore \phi = \frac{x^2}{2} + 2xy + 4xz - zy - \frac{3y^2}{2} + z^2$$

where $f(y,z) = -zy - \frac{3}{2}y^2 + z^2$

$h(x,z) = \frac{x^2}{2} + 4xz + z^2$

$g(x,y) = \frac{x^2}{2} + 2xy - \frac{3}{2}y^2$.

2. By Green's Theorem,

$$\oint_C (x^2+y^2)\,dx + (3xy^2)\,dy$$

$$= \iint \frac{\partial}{\partial x}(3xy^2) - \frac{\partial}{\partial y}(x^2+y^2)\,dx\,dy$$

$$= \iint 3y^2 - 2y\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^2 (3r^2\sin^2\theta - 2r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 (3r^3\sin^2\theta - 2r^2\sin\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{3}{4}r^4\sin^2\theta - \frac{2}{3}r^3\sin\theta \Big|_0^2\,d\theta$$

$$= \int_0^{2\pi} 12\sin^2\theta - \frac{16}{3}\sin\theta\,d\theta$$

Double angle
$2\sin^2\theta = 1 - \cos 2\theta$
$\therefore 12\sin^2\theta = 6 - 6\cos 2\theta$

$$= \int_0^{2\pi} 6 - 6\cos 2\theta - \frac{16}{3}\sin\theta\,d\theta$$

$$= 6\theta - 3\sin 2\theta + \frac{16}{3}\cos\theta \Big|_0^{2\pi}$$

$$= \left(12\pi - 3\sin 4\pi + \frac{16}{3}\cos 2\pi\right) - \left(0 - 3\sin 0 + \frac{16}{3}\cos 0\right)$$

$$= 12\pi + \frac{16}{3} - \frac{16}{3}$$

$$= \underline{12\pi}$$

3. The transformations for spherical coordinates are:

$$x = \rho \cos\theta \sin\phi$$
$$y = \rho \sin\theta \sin\phi$$
$$z = \rho \cos\phi$$

where $\rho \geq 0$, $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$

$\therefore$ In spherical coordinates $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

becomes $\bar{r} = \rho\cos\theta\sin\phi\,\hat{i} + \rho\sin\theta\sin\phi\,\hat{j} + \rho\cos\phi\,\hat{k}$.

We know that the Volume element dV is given by $dV = h_\rho h_\theta h_\phi \, d\rho\, d\theta\, d\phi$ ... (*)

where the scale factors h_ρ, h_θ and h_ϕ are determined as follows

$$h_\rho = \left| \frac{\partial \bar{r}}{\partial \rho} \right| = \left| \cos\theta\sin\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\phi\,\hat{k} \right|$$

$$= \sqrt{\cos^2\theta\sin^2\phi + \sin^2\theta\sin^2\phi + \cos^2\phi}$$

$$= \sqrt{\sin^2\phi(\cos^2\theta + \sin^2\theta) + \cos^2\phi}$$

$$= \sqrt{\sin^2\phi + \cos^2\phi}$$

$$= 1$$

Similarly $h_\alpha = \left|\frac{\partial \bar{r}}{\partial \alpha}\right| = \left| -\rho \sin\alpha \sin\phi \hat{i} + \rho \cos\alpha \sin\phi \hat{j} \right|$

$$= \sqrt{\rho^2 \sin^2\alpha \sin^2\phi + \rho^2 \cos^2\alpha \sin^2\phi}$$

$$= \rho \sin\phi \sqrt{\sin^2\alpha + \cos^2\alpha}$$

$$= \rho \sin\phi$$

and, $h_\phi = \left|\frac{\partial \bar{r}}{\partial \phi}\right|$

$$= \left| \rho \cos\alpha \cos\phi \hat{i} + \rho \sin\alpha \cos\phi \hat{j} - \rho \sin\phi \hat{k} \right|$$

$$= \sqrt{\rho^2 \cos^2\alpha \cos^2\phi + \rho^2 \sin^2\alpha \cos^2\phi - \rho^2 \sin^2\phi}$$

$$= \rho \sqrt{\cos^2\phi (\cos^2\alpha + \sin^2\alpha) + \sin^2\phi}$$

$$= \rho \sqrt{\cos^2\phi + \sin^2\phi}$$

$$= \rho$$

Substituting the scale factors into ⊛, the volume element dV in spherical coordinates is given by $dV = 1 . \rho \sin\phi . \rho \; d\rho \, d\alpha \, d\phi$

$$\therefore \quad \underline{dV = \rho^2 \sin\phi \; d\rho \, d\alpha \, d\phi}$$

Now, the volume inside the sphere of radius r is given by $\iiint_V dV$ where

$$\iiint_V dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{r} \rho^2 \sin\phi \, d\rho \, d\alpha \, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi} \frac{\rho^3}{3} \sin\phi \Big|_0^r \, d\alpha \, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi} \frac{r^3}{3} \sin\phi \, d\alpha \, d\phi$$

$$= \int_0^{\pi} \alpha \cdot \frac{r^3}{3} \sin\phi \Big|_0^{2\pi} \, d\phi$$

$$= \int_0^{\pi} 2\pi \frac{r^3}{3} \sin\phi \, d\phi$$

$$= \frac{2\pi r^3}{3} \left(- \cos\phi \right) \Big|_0^{\pi}$$

$$= \frac{2\pi r^3}{3} \left(-\cos\pi + \cos 0 \right)$$

$$= \frac{2\pi r^3}{3} (2)$$

$$= \frac{4}{3}\pi r^3$$ as required.